Stochastic transient of a noise-perturbed Haken-Zwanzig model

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A bivariate nonlinear model perturbed by external white noises is investigated stochastically. Attention is concentrated on the transient properties before the nonequilibrium phase is achieved. Effects of both additive and multiplicative noises are found to weaken stability and to slow down transient processes. The critical exponent describing this slowing-down phenomenon near a noise-induced instability is estimated for various types of noises. Results derived with two versions of stochastic calculus are compared systematically.

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I. INTRODUCTION

Noise-affected dynamics have been one of the most important topics in nonequilibrium physics [1,2]. Noises are unavoidable for open systems which are subject to external constraints and perturbations [3]. Studies of noise effects could enable us to stabilize the operation of nonequilibrium systems. On the other hand, we could gain more insight into the transition phenomena which could not be found in equilibrium counterparts [1,4–6].

Steady-state (SS) properties of noisy systems have attracted most of the attention [1–3,7,8]. Transient processes, which are a unique part of nonequilibrium physics, deserve more effort. Noise problems are usually difficult ones since ordinary Riemann calculus would fail in treating the noise parameters which fluctuate randomly and rapidly in time [1]. Most of the studies have been limited to one-dimensional systems. Approximate and concise stochastic transients in both one- and two-dimensional systems can be derived by employing a moment-expansion scheme [4–6].

Critical slowing down, like other dynamic critical phenomena in equilibrium systems, is well documented, and is properly described by statistical mechanical theories [9]. In nonequilibrium systems, however, results are fragmented and conclusions are controversial [4,6,10–12]. It is reasonable to believe that external noises disturb the transient processes and slow down the relaxation. More studies on stochastic transients near a noise-induced instability are needed.

In this report, we treat a bivariate Haken-Zwanzig model [13,14], which was used to demonstrate the slaving principle in *synergetics* and is related to the more complex *ABCDE* model [15]. The model is described by two order parameters which satisfy a set of coupled rate equations,

$$\frac{du}{dt}=f_1(u,s)=\epsilon u-us\ ,$$
$$\frac{ds}{dt}=f_2(u,s)=-\gamma s+u^2\ , \tag{1}$$

where rate parameters (ϵ,γ) are nonnegative, and so are (u,s).

In Sec. II we discuss the transient properties described by the deterministic rate equations, and those described with considerations of internal fluctuations. White noise formulations are developed in Sec. III. Noise-affected transients due to additive and multiplicative noises are treated in Sec. IV. In Sec. V we present a unique phenomenon of critical slowing down near a noise-induced instability. Results are discussed in Sec. VI.

II. DETERMINISTIC TRANSIENT AND INTERNAL FLUCTUATIONS

Analytical solution of $u(t)$ and $s(t)$ from Eq. (1) is not feasible; numerical method is required. A complete description of the solution involves assignments of values to the two rate parameters (ϵ,γ) and the two initial conditions (u_0,s_0).

Unlike the one studied previously [5], this two-dimensional model is stable and has two fixed points which are given as SS solutions of Eq. (1),

$$(u_s,s_s)_{\rm I}=(0,0)\ ,$$
$$(u_s,s_s)_{\rm II}=(\sqrt{\epsilon\gamma},\epsilon)\ . \tag{2}$$

The first SS describes the extinction of dynamic process, and is unstable for the choice of $\epsilon>0$ and $\gamma>0$. The second one describes the coexistence of two modes (u,s), and is stable since the stability eigenvalue is given by

$$\lambda=-(\gamma/2)[1\pm\sqrt{1-8\epsilon/\gamma}\,]\ . \tag{3}$$

Therefore a nontrivial system at any initial state will relax toward the coexisting SS. While the stability of this SS depends strongly on the value of γ, the time scale characterizing the transient is found to be depending mainly on the parameter ϵ. Figure 1 demonstrates two typical transients of deterministic $u(t)$ and $s(t)$.

Equation (3) predicts that restoration of the SS from any small deviation will show an oscillatory pattern if $\gamma<8\epsilon$. Numerical results seem to suggest that this conclusion is also applicable to a global relaxation from an initial state which could be very far away from equilibrium.

The time required to achieve a SS is infinite in mathematical sense. This is true for both monotonic and oscillatory relaxations. Technically, we can define the *relaxation time* as the time interval for a system to relax toward the SS within an accuracy of, say, one part in 10^6. This criterion, or the other, would reflect the computation or measurement resolution. It is found that the relaxation time depends sensitively on the values of ϵ and γ, and is roughly the same for typical choices of initial conditions $u_0/u_s \sim s_0/s_s \sim 10^{-1}$.

The dynamic system should be described by a probability function $P(u,s;t)$ if internal fluctuations are taken into consideration. For Markovian processes with single-step variations in u and s, $P(u,s;t)$ is found to satisfy a master equation which can be written as

$$\partial_t P(u,s;t)=\epsilon(u-1)P(u-1,s,t)+s(u+1)P(u+1,s;t)+\gamma(s+1)P(u,s+1;t) \\ +u^2P(u,s-1;t)-[\epsilon u+su+\gamma u+u^2]P(u,s;t) \; . \tag{4}$$

Analytical solution of $P(u,s;t)$ is unlikely for this differential-difference equation.

In Gaussian approximation, $P(u,s;t)$ can be approximated by having its peak located at the mean values of order parameters $(\bar{u},\bar{s})$, and having its shape determined by the three variances σ_{uu}, σ_{ss}, and σ_{us}, where

$$\sigma_{ij}=\overline{x_i x_j}-\bar{x}_i\bar{x}_j \; . \tag{5}$$

With the moment-expansion scheme [4], the moment equations derived from the master equation (4) can be expressed as

$$\begin{aligned} &\frac{d\bar{u}}{dt}=\epsilon\bar{u}-\bar{s}\,\bar{u}-\sigma_{us} \; , \quad \frac{d\bar{s}}{dt}=-\gamma\bar{s}+\bar{u}^2+\sigma_{uu} \; , \\ &\frac{d\sigma_{uu}}{dt}=2(\epsilon\sigma_{uu}-\bar{s}\sigma_{uu}-\bar{u}\sigma_{us})+2\epsilon\bar{u}-\frac{d\bar{u}}{dt} \; , \\ &\frac{d\sigma_{ss}}{dt}=-2\gamma\sigma_{ss}+4\bar{u}\sigma_{us}+2\gamma\bar{s}-\frac{d\bar{s}}{dt} \; , \\ &\frac{d\sigma_{us}}{dt}=(\epsilon-\gamma-\bar{s})\sigma_{us}-\bar{u}\sigma_{ss}+2\bar{u}\sigma_{uu} \; . \end{aligned} \tag{6}$$

Numerical treatment is required to solve these coupled equations. Internal fluctuations usually introduce small relative fluctuations,

$$R_i(t)=[\sigma_{ii}(t)]^{1/2}/\bar{x}_i(t) \; , \tag{7}$$

to a system with large value of order parameter. System-size effect may appear if (u,s) are small; this will happen if (ϵ,γ) are assigned to small values. Figure 1 demonstrates further that during transit, $u(t)$ and $s(t)$ may drop to values so small that size effect comes into play and the assumption of single-step processes fails.

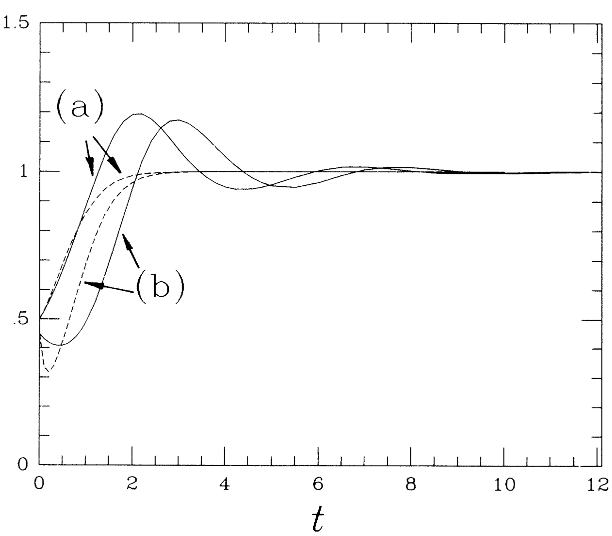

FIG. 1. Deterministic evolution of order parameters $u(t)/u_s$ and $s(t)/s_s$ is represented by curves (a) and (b), respectively. Solid curves are for $(\epsilon,\gamma)=(1,1)$, and dashed ones are for $(\epsilon,\gamma)=(1,10)$.

III. WHITE NOISE FORMULATIONS

External fluctuations may appear as additive random forcing or as variations in rate parameters. The deterministic rate equations (1) should be rewritten as stochastic differential equations,

$$\begin{aligned} &\frac{du}{dt}=f_1(u,s)+D_1g_1(u,s)\xi_1(t) \; , \\ &\frac{ds}{dt}=f_2(u,s)+D_2g_2(u,s)\xi_2(t) \; , \end{aligned} \tag{8}$$

where the multiplication function $g_i(u,s)$ describes the state dependence of noises. For noises in rate parameters ϵ and γ, we have, respectively, $g_1=u$ and $g_2=-s$. For additive noises, we have $g_1=g_2=1$. In Eq. (8), D_i describes the amplitude of the fluctuating quantity,

$$\lambda_{it}=\lambda_i+D_i\xi_{it} \; , \tag{9}$$

where λ_i is the averaged value of rate parameter (for the case of additive noises, $\lambda_i=0$). The random and rapid fluctuating variables ξ_{it} are assumed to have zero mean, and are δ-function correlated,

$$\langle \xi_{it} \rangle=0 \; , \quad \langle \xi_{it}\xi_{jt'} \rangle=\delta_{ij}\delta(t-t') \; . \tag{10}$$

The probability function $P(u,s;t)$ is found to satisfy a Fokker-Planck equation, which may assume a different form for a different stochastic calculus. For the two popular interpretations, we have

$$\partial_t P=-\partial_u[f_1+(\nu-1)D_1^2g_1\partial_u g_1/2]P+D_1^2\partial_{uu}[g_1^2P]/2 \\ -\partial_s[f_2+(\nu-1)D_2^2g_2\partial_s g_2/2]P+D_2^2\partial_{ss}[g_2^2P]/2 \; , \tag{11}$$

where $\nu=1$ and 2 stand, respectively, for the case of Ito and Stratonovich interpretations.

The stochastic transient is described by the moment equations derived from Fokker-Planck equation (11),

$$\frac{d\overline{u^m s^n}}{dt} = m\overline{u^{m-1}s^n f_1} + n\overline{u^m s^{n-1} f_2} + (\nu-1)[D_1^2 m\overline{u^{m-1}s^n g_1 \partial_u g_1} + D_2^2 n\overline{u^m s^{n-1} g_2 \partial_s g_2}]/2 + \frac{D_1^2 m(m-1)\overline{u^{m-2}s^n g_1^2} + D_2^2 n(n-1)\overline{u^m s^{n-2} g_2^2}}{2} . \quad (12)$$

These coupled equations of infinite hierarchy can be truncated by using the moment-expansion scheme. The resulting moment equations depend on the g_i functions which specify the type of noises.

IV. NOISE-PERTURBED TRANSIENTS

For the simple case of additive noises, both Ito and Stratonovich calculus result in the same form of five moment equations,

$$\frac{d\bar{u}}{dt} = \bar{u} - \bar{u}\,\bar{s} - \sigma_{us} , \quad \frac{d\bar{s}}{dt} = -\gamma\bar{s} + \bar{u}^2 + \sigma_{uu} ,$$
$$\frac{d\sigma_{uu}}{dt} = 2(\epsilon\sigma_{uu} - \bar{u}\sigma_{us} - \bar{s}\sigma_{uu}) + D_1^2 ,$$
$$\frac{d\sigma_{ss}}{dt} = -2\gamma\sigma_{ss} + 4\bar{u}\sigma_{us} + D_2^2 , \quad (13)$$
$$\frac{d\sigma_{us}}{dt} = (\epsilon - \gamma - \bar{s})\sigma_{us} - \bar{u}\sigma_{ss} + 2\bar{u}\sigma_{uu} .$$

This set of equations allows us to treat three types of additive noises: those which occur in either one of the two modes and on both modes at the same time. This can be done by assigning appropriate values to D_i. For a concise presentation, we shall report results only for $D_1 = D_2$.

For a moderate value of noise intensity, evolutions of $u(t)$ and $s(t)$ are very close to those of deterministic ones with insignificant variances. This conclusion applies also to the SS properties. We plot a typical result in Fig. 2.

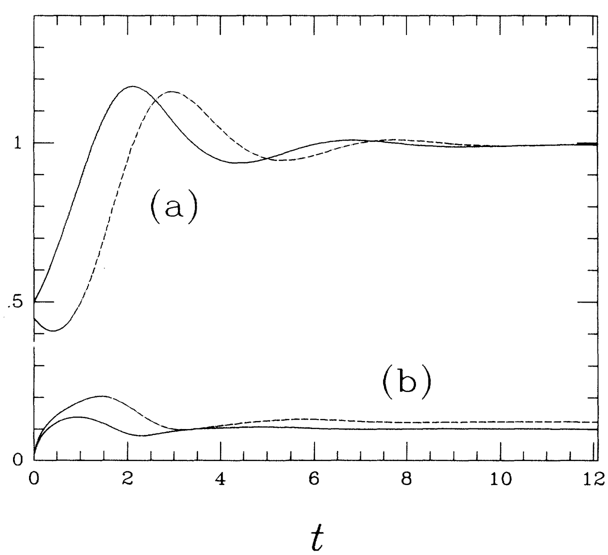

FIG. 2. Transient towards a coexisting SS of a system perturbed by additive noises with $D_1 = D_2 = 0.1$. Mean values (a) and relative fluctuations (b) are plotted against time. Solid and dashed curves stand, respectively, for the first (u) and second (s) modes. $(\epsilon,\gamma) = (1,1)$ is assumed.

External perturbations may appear as variations in rate parameters. Influence of noise becomes state dependent since the g_i functions are no longer constant. There are also three distinct types of multiplicative noises: those on ϵ ($g_1 = u$, $g_2 = 0$), or on γ ($g_1 = 0$, $g_2 = -s$), or on both of them ($g_1 = u$, $g_2 = -s$). Again we shall consider only the third type and assume that $D_\epsilon = D_\gamma$. The moment equations for both Ito and Stratonovich interpretations can be written as

$$\frac{d\bar{u}}{dt} = [\epsilon + (\nu-1)D_\epsilon^2/2]\bar{u} - \bar{u}\,\bar{s} - \sigma_{us} ,$$
$$\frac{d\bar{s}}{dt} = -[\gamma - (\nu-1)D_\gamma^2/2]\bar{s} + \bar{u}^2 + \sigma_{uu} ,$$
$$\frac{d\sigma_{uu}}{dt} = 2(\epsilon\sigma_{uu} - \bar{u}\sigma_{us} - \bar{s}\sigma_{uu}) + D_\epsilon^2(\nu\sigma_{uu} + \bar{u}^2) , \quad (14)$$
$$\frac{d\sigma_{ss}}{dt} = -2\gamma\sigma_{ss} + 4\bar{u}\sigma_{us} + D_\gamma^2(\nu\sigma_{ss} + \bar{s}^2) ,$$
$$\frac{d\sigma_{us}}{dt} = [\epsilon - \gamma - s + (\nu-1)(D_\epsilon^2 + D_\gamma^2)/2]\sigma_{us} - \bar{u}\sigma_{ss} + 2\bar{u}\sigma_{uu} .$$

Again, for small and moderate values of D_i, evolution of the system is more or less deterministic. The SS mean values of order parameters are slightly smaller than the deterministic values, while the relative fluctuations R_i remain small. Only when noise in rate parameters, D_i, becomes comparable to the averaged mean do stochastic aspects become significant.

Figure 3 shows the noise-intensity dependence of mean values and relative fluctuations. Results are derived from the Stratonovich interpretation, and are found to have little difference from those derived from the Ito interpretation and from those for additive noises. All three sets of results predict the existence of a noise-induced instability. At this unstable point, mean values $(\bar{u}_s, \bar{s}_s)$ drop rapidly while relative fluctuations diverge. This stochastic behavior is represented by a vertical line in Fig. 3. It is interesting to note that both additive and multiplicative noises (in either one of the two representations) predict the same behaviors. Quantitatively, they predict similar values for the critical D at which the SS system becomes unstable,

$$D_c^A = 0.373 , \quad D_c^S = 0.400 , \quad D_c^I = 0.409 . \quad (15)$$

In the above, superscripts A, S, and I stand, respectively, for additive noise, and multiplicative noises in the Stratonovich and Ito interpretations.

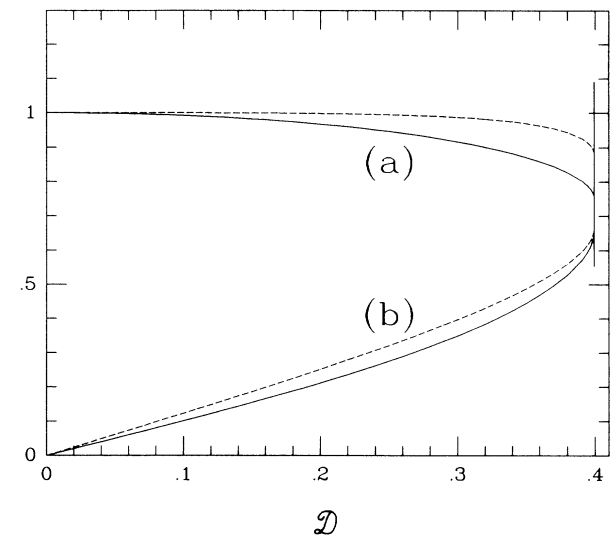

FIG. 3. SS values of scaled means (a) and relative fluctuations (b) are plotted against the noise intensity. Solid and dashed curves stand, respectively, for the first (u) and second (s) modes. Multiplicative noises in the Stratonovich interpretation, and $(\epsilon,\gamma)=(1,1)$ are assumed.

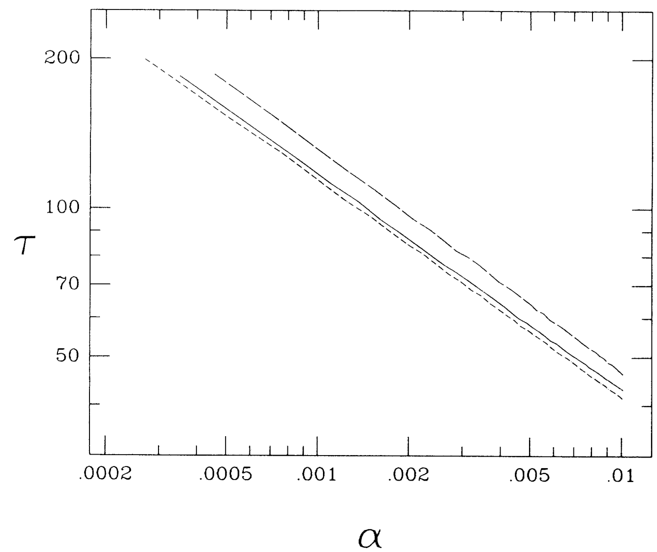

FIG. 5. Critical slowing down is demonstrated by plotting the relaxation time (τ) against the noise intensity (α). Results due to additive and multiplicative noises in the Stratonovich and Ito interpretations are plotted, respectively, by dashed, solid, and dotted curves.

For D larger than the critical value D_c, the coexistence phase transits to the extinction phase at which both modes are subject to fluctuation catastrophe. This noise-induced transition is purely stochastic in nature [4,5].

V. CRITICAL SLOWING DOWN

The narrow region preceding the instability can be defined as a *critical region,* since the phenomena of abrupt changes within this region resemble *critical phenomena* of equilibrium systems. Detailed numerical investigations show that relaxation times t_s also increases abruptly when the noise-induced instability is approached. This is shown in Fig. 4 for three different sets of results.

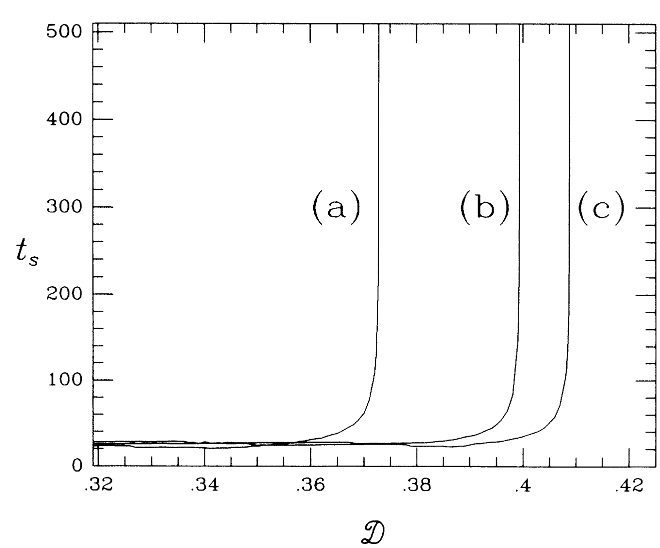

FIG. 4. Relaxation times are plotted against the noise intensity. Results due to additive (a), and multiplicative noises in the Stratonovich (b) and Ito (c) interpretations are presented for comparison.

The criticality can be better described by defining reduced parameters as,

$$\tau=t_s\ ,\quad \alpha=(D_c-D)/D_c\ . \tag{16}$$

Numerical results show that the divergence of τ follows a simple power law,

$$\tau\propto\alpha^{-\Gamma}\ , \tag{17}$$

where the exponent Γ describes the *critical slowing-down* phenomenon. It is estimated for the three sets of data as

$$\Gamma^A=0.449\ ,\quad \Gamma^S=0.434\ ,\quad \Gamma^I=0.437\ . \tag{18}$$

It is interesting to see that we obtain roughly the same critical exponent for these sets of data, as is shown in Fig. 5. In previous studies of one-dimensional systems of both quadratic and cubic nonlinearities [4], we obtain the critical exponent $\Gamma=1.0$.

VI. DISCUSSIONS AND CONCLUSIONS

Noises are found to have destructive effects on dynamic systems. They raise fluctuations and destabilize the SS. In studies of the present two-dimensional system and the previous one-dimensional ones [4], we are able to estimate the scope of the noise effects on the transient as well as on the SS properties. It is found that noise effects depend on types of noises, i.e., on the structures of the g function. Usually additive noises cause smaller effects than the multiplicative ones might. In general, effects are insignificant for small and moderate values of noises.

Studies on noise effects enable us to probe the stochastic properties of noise-induced transitions which are unique only to the nonequilibrium systems. More investigation is needed in order to understand the nature of phase transition and critical phenomena in nonequilibrium systems, especially the controversial issue of critical slowing down [4,6,10–12].

Both theoretical and experimental studies tend to favor the usage of the Stratonovich interpretation in stochastic formulation of physical systems [16–18]. We present here both Stratonovich and Ito results for comparison. For the present studies, they predict not only qualitatively the same transient properties, but also quantitatively the same exponent for the critical slowing-down phenomenon.

Analysis of transient properties relies on approximation schemes. Our results are reliable, since fluctuations at SS and during transient are relatively small. Moment-expansion approximation may fail if the relative fluctuations approach the order of unity. The fluctuation catastrophe at the noise-induced instability and the phase transition afterward cannot be described in our analysis. They are only qualitatively predicted here.

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